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The Holonomic Rank of the Fisher-Bingham System of Differential Equations

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Abstract. The Fisher-Bingham system is a system of linear partial differential equations satisfied by the Fisher-Bingham integral for the n-dimensional sphere S^n. The system is given in [4, Theorem 2] and it is shown that it is a holonomic system [1]. We show that the holonomic rank of the system is equal to $2n+2$.

Keywords: Fisher-Bingham distribution, holonomic rank, Gröbner basis

1 Introduction

Let $x = (x_{ij})$ and $y = (y_i)$ be parameters such that $x_{ij} = x_{ji}$ for $i \neq j$. Let Z be a function, which is the normalization constant of the Fisher-Bingham distribution, defined as

$$Z(x, y, r) = \int_{S^n(r)} \exp\left(\sum_{1 \leq i \leq j \leq n+1} x_{ij} t_i t_j + \sum_{i=1}^{n+1} y_i t_i \right) |dt| \tag{1}$$

where $S^n(r) = \{(t_1, \ldots, t_{n+1}) \mid \sum_{i=1}^{n+1} t_i^2 = r^2, r > 0\}$ is the n-dimensional sphere and $|dt|$ denotes the Haar measure on the sphere.

Let D be the Weyl algebra

$$D = \mathbf{C}\langle x_{ij}, y_k, r, \partial_{ij}, \partial_k, \partial_r \mid 1 \leq i \leq j \leq n+1, 1 \leq k \leq n+1 \rangle$$

where $\partial_{ij} = \partial/\partial x_{ij}$, $\partial_k = \partial/\partial y_k$ and $\partial_r = \partial/\partial r$. It is shown in [1] and [4] that the normalization constant (1) of the Fisher-Bingham distribution is a holonomic function in x, y, r and consequently it is annihilated by the following

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holonomic ideal I in D generated by the following operators in D:

$$\begin{aligned}
&\partial_{ij} - \partial_i \partial_j, \\
&\sum_{i=1}^{n+1} \partial_i^2 - r^2, \\
&x_{ij}\partial_i^2 + 2(x_{jj} - x_{ii})\partial_i\partial_j - x_{ij}\partial_j^2 \\
&\qquad + \sum_{s \neq i,j} (x_{sj}\partial_i\partial_s - x_{is}\partial_j\partial_s) + y_j\partial_i - y_i\partial_j, \\
&r\partial_r - 2\sum_{i \leq j} x_{ij}\partial_i\partial_j - \sum_i y_i\partial_i - n.
\end{aligned}$$

We call the system of differential equations defined by I *the Fisher-Bingham system*.

For a left ideal J in D, the holonomic rank of J is defined as the dimension of the $K = \mathbf{C}(x_{ij}, y_k, r \mid 1 \leq i \leq j \leq n+1, 1 \leq k \leq n+1)$ vector space $K\langle \partial_{ij}, \partial_k, \partial_r \rangle / K\langle \partial_{ij}, \partial_k, \partial_r \rangle J$. The rank is denoted by $\mathrm{rank}\,(J)$. When J is a holonomic ideal, the rank is finite. The holonomic rank agrees with the dimension of the holomorphic solutions of the associated system of linear partial differential equations at generic points and with the size of the Pfaffian equation associated to J. As to general facts on the holonomic rank, we refer to, e.g., the chapters 1 and 2 of [6]. The holonomic rank is a fundamental invariant of the D-module D/J and there are several attractive studies on holonomic ranks. For example, Miller, Matusevich and Walther studied holonomic ranks of A-hypergeometric systems by introducing a new homological method [3].

We are interested in the holonomic rank of the Fisher-Bingham system I. We prove the following theorem in this paper.

Theorem 1

$$\mathrm{rank}\,(I) = 2n + 2$$

In [4], we proposed a new method in the statistical inference which is called the holonomic gradient descent. The method utilizes a holonomic system of linear partial differential equations associated to the normalization constant. The complexity of the method depends on the holonomic rank and correctness of the method are proved by utilizing the holonomic rank. In the case of the Fisher-Bingham distribution, which is the most fundamental distribution in the directional statistics, Theorem 1 is applied in [2], which gives a generalization of the result in [4] shown with a help of a computer program. Our method to prove the theorem is the Gröbner deformation to the direction $(-w, w)$, which is discussed in [6] for A-hypergeometric systems, and a determination of Gröbner bases *by hand* with adding several *slack variables* which do not change the holonomic rank.

2 The Rank of the Diagonal System

When the matrix x is diagonal, the normalization constant Z satisfies a system of linear partial differential equations for the variables x_{ii}, y_k, r. Let $\tilde{I}$ be the left ideal in D generated by

$$\begin{aligned}
A_i &= \partial_{ii} - \partial_i^2 \quad (1 \le i \le n+1),\\
B &= \sum_{i=1}^{n+1} \partial_i^2 - r^2,\\
C_{ij} &= 2(x_{ii} - x_{jj})\partial_i\partial_j + y_i\partial_j - y_j\partial_i \quad (1 \le i < j \le n+1),\\
E &= r\partial_r - 2\sum_{i=1}^{n+1} x_{ii}\partial_i^2 - \sum_{i=1}^{n+1} y_i\partial_i - n
\end{aligned}$$

and ∂_{ij}, $i \ne j$. The ideal $\tilde{I}$ annihilates the function Z restricted to the diagonal of x [2].

Theorem 2 *The holonomic rank of $\tilde{I}$ is $2n+2$.*

Our proof of the theorem reduces to the proof of the following proposition.

Proposition 1 *Let R be the ring of differential operators with rational function coefficients*

$$R = \mathbf{C}(x_{11}, \ldots, x_{n+1n+1}, y_1, \ldots, y_{n+1}, r)\langle \partial_{11}, \ldots, \partial_{n+1n+1}, \partial_1, \ldots, \partial_{n+1}, \partial_r \rangle.$$

Let $R\tilde{I}$ be the left ideal of R generated by A_i, B, C_{ij}, E. Let $<$ be the term order on R which is the block order with $\partial_r \gg \{\partial_{ii}\} \gg \{\partial_j\}$. The order of the block $\{\partial_{ii}\}$ is the graded lexicographic order with $\partial_{11} > \cdots > \partial_{n+1n+1}$ and that of the block $\{\partial_i\}$ is the graded lexicographic order with $\partial_1 > \cdots > \partial_{n+1}$. A Gröbner basis of $R\tilde{I}$ with respect to the term order $<$ is

$$\begin{aligned}
A_i &= \partial_{ii} - \partial_i^2 \quad (i = 1, \ldots, n+1),\\
B &= \sum_{i=1}^{n+1} \partial_i^2 - r^2,\\
C_{ij} &= 2(x_{ii} - x_{jj})\partial_i\partial_j + y_i\partial_j - y_j\partial_i \quad (1 \le i < j \le n+1),\\
&(\textit{We put } a_{ij} = 2(x_{ii} - x_{jj}), F_{ij} = y_i\partial_j - y_j\partial_i, C_{ij} = a_{ij}\partial_i\partial_j + F_{ij}),\\
D_k &= \partial_k B - \partial_1 a_{1k}^{-1} C_{1k} - \cdots - \partial_{k-1} a_{k-1k}^{-1} C_{k-1k} \quad (k = 1, \ldots, n+1),\\
E &= r\partial_r - 2\sum_{i=1}^{n+1} x_{ii}\partial_i^2 - \sum_{i=1}^{n+1} y_i\partial_i - n.
\end{aligned}$$

The initial monomials of the Gröbner basis are

$$\begin{aligned}
&\mathrm{in}_<(A_i) = \mathrm{in}_<(\partial_{ii}), \mathrm{in}_<(B) = \mathrm{in}_<(\partial_1)^2, \mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j),\\
&\mathrm{in}_<(D_k) = \mathrm{in}_<(\partial_k)^3, \mathrm{in}_<(E) = \mathrm{in}_<(\partial_r).
\end{aligned}$$

Here, the initial monomial $\mathrm{in}_<(c(x,y,r)\partial_r^\gamma \prod \partial_{ii}^{\alpha_{ii}} \prod \partial_k^{\beta_k})$ is defined as the element $c(x,y,r)\iota^\gamma \prod \xi_{ii}^{\alpha_{ii}} \prod \eta_k^{\beta_k}$ in the polynomial ring with rational function coefficients $\mathbf{C}(x,y,r)[\xi_{11},\ldots,\xi_{n+1n+1},\eta_1,\ldots,\eta_{n+1},\iota]$ (see, e.g., [6, Chapter 1]).

By the proposition, the standard monomials of the quotient ring $R/R\tilde{I}$ are $1,\partial_1,\partial_2,\partial_2^2,\ldots,\partial_{n+1},\partial_{n+1}^2$. Therefore, the holonomic rank of $\tilde{I}$ is $2n+2$ (Theorem 2). Let us prove the proposition.

Since D_k is expressed by $B, C_{1k},\ldots,C_{k-1k}$, the operator D_k is the element in $R\tilde{I}$. In order to prove Proposition 1, we will show that any S-pair for A_i, B, C_{ij}, D_k, E is reduced to 0 by A_i, B, C_{ij}, D_k, E. The following lemmas are proved by straight forward calculations.

Lemma 1 *Let P and Q be elements in R. If the initial monomials are coprime, i.e., $\gcd(\mathrm{in}_<(P),\mathrm{in}_<(Q))=1$, then the S-pair $S(P,Q)$ is reduced to $[P,Q]$ by P and Q (we denote the reduction by $S(P,Q) \xrightarrow[P,Q]{}^* [P,Q]$), where $[P,Q]$ is the commutator of P and Q. In particular, when $[P,Q]=0$, the S-pair $S(P,Q)$ is reduced to 0.*

Lemma 2 *We have*

$$[A_p, C_{ij}] = 0,$$
$$[B, C_{ij}] = 0,$$
$$[C_{ij}, C_{jk}] = C_{ik}, [C_{ij}, C_{ik}] = -C_{jk}, [C_{ik}, C_{jk}] = -C_{ij} \quad (i<j<k),$$
$$[C_{ij}, C_{pq}] = 0 \quad (\{i,j\}\cap\{p,q\}=\emptyset).$$

Lemma 3 *We have*

$$[D_i, A_j] = \begin{cases} 0 & (i<j) \\ 2a_{ji}^{-2}\partial_j C_{ji} & (i>j), \\ \sum_{l=1}^{i-1} 2a_{li}^{-2}\partial_l C_{li} & (i=j) \end{cases}$$
$$[D_i, B] = 0,$$
$$[D_i, D_j] = -B\partial_j + \sum_{l<i} a_{li}^{-1}a_{ij}^{-1}\partial_l(\partial_i C_{lj} + \partial_l C_{ij}) + \sum_{l<i} a_{li}^{-1}a_{lj}^{-1}(-2\partial_l C_{ij}).$$

When $i,j \neq k$, we obtain

$$[C_{ij}, D_k] = \begin{cases} 0 & (k-1<i) \\ 0 & (j \le k-1) \\ -a_{ik}^{-1}[C_{ij}, \partial_i C_{ik}] = a_{ik}^{-1}(\partial_i C_{jk} + \partial_j C_{ik}) & (i \le k-1 < j) \end{cases}.$$

Lemma 4 *We have*

$$[A_i, E] = 0,$$
$$[B, E] = -2B,$$
$$[C_{ij}, E] = 0,$$
$$[D_i, E] = -3D_i - 2\sum_{k=1}^{i-1} a_{ki}^{-1}\partial_k C_{ki}.$$

Proof of Proposition 1. We prove that any S-pair for A_i, B, C_{ij}, D_k, E is reduced to 0 by A_i, B, C_{ij}, D_k, E.

S-pairs of A_i and A_j, B, C_{ij}. The initial monomials are coprime, and the elements commute. By Lemma 1, we obtain

$$\begin{aligned}
&S(A_i, A_j) \longrightarrow^* 0,\\
&S(A_i, B) \longrightarrow^* 0,\\
&S(A_i, C_{jk}) \longrightarrow^* 0.
\end{aligned}$$

S-pair of B and C_{ij}. When $i > 1$, the initial monomials $\mathrm{in}_<(B) = \mathrm{in}_<(\partial_1)^2$, $\mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j)$ are coprime. Operators B and C_{ij} commute by Lemma 2. By Lemma 1, we have $S(B, C_{ij}) \longrightarrow^* 0$.

When $i = 1$, the initial monomials $\mathrm{in}_<(B) = \mathrm{in}_<(\partial_1)^2$, $\mathrm{in}_<(C_{1j}) = \mathrm{in}_<(\partial_1)\mathrm{in}_<(\partial_j)$ are not coprime. We obtain the following reduction sequence of the S-pair:

$$\begin{aligned}
&S(B, C_{1j}) = a_{1j}\partial_j B - \partial_1 C_{1j}\\
&= a_{1j}(\partial_j\partial_2^2 + \cdots + \partial_j^3 + \cdots + \partial_j\partial_{n+1}^2 - \partial_j r^2) - \partial_1 F_{1j}\\
&= a_{1j}((\underline{\partial_j\partial_2^2} + \cdots + \partial_j^3 + \cdots + \partial_j\partial_{n+1}^2 - \partial_j r^2) - \partial_1 a_{1j}^{-1} F_{1j})\\
&\xrightarrow[C_{2j}]{}^* a_{1j}((\underline{\partial_j\partial_3^2} + \cdots + \partial_j^3 + \cdots + \partial_j\partial_{n+1}^2 - \partial_j r^2) - \partial_1 a_{1j}^{-1} F_{1j} - \partial_2 a_{2j}^{-1} F_{2j})\\
&\xrightarrow[C_{3j}]{}^* \cdots \xrightarrow[C_{j-1j}]{}^* a_{1j} D_j \xrightarrow[D_j]{} 0.
\end{aligned}$$

S-pair of C_{ij} and C_{kl} ($\{i, j\} \cap \{k, l\} = \emptyset$). The initial monomials $\mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j), \mathrm{in}_<(C_{kl}) = \mathrm{in}_<(\partial_k)\mathrm{in}_<(\partial_l)$ are coprime. Operators C_{ij} and C_{kl} commute by Lemma 2. By Lemma 1, we obtain

$$S(C_{ij}, C_{kl}) \xrightarrow[C_{ij}, C_{kl}]{}^* 0.$$

S-pair of C_{ij} and C_{jk} ($i < j < k$). We have the following reduction sequence of the S-pair:

$$\begin{aligned}
S(C_{ij}, C_{jk}) &= a_{jk}\partial_k C_{ij} - a_{ij}\partial_i C_{jk} = -a_{ik} y_j \partial_i\partial_k + a_{jk} y_i \partial_j\partial_k + a_{ij} y_k \partial_i\partial_j\\
&\xrightarrow[C_{ij}, C_{jk}, C_{kl}]{}^* y_i(-F_{jk}) + y_k(-F_{ij}) - y_j(-F_{ik}) = 0.
\end{aligned}$$

The S-pair of C_{ij} and C_{ik} and that of C_{ik} and C_{jk} are also reduced to 0.

S-pair of D_i and A_j. The initial monomials $\mathrm{in}_<(D_i) = \mathrm{in}_<(\partial_i)^3, \mathrm{in}_<(A_j) = \mathrm{in}_<(\partial_{jj})$ are coprime. When $i < j$, operators D_i and A_j commute. By Lemma 1, we have

$$S(D_i, A_j) \xrightarrow[D_i, A_j]{}^* 0.$$

When $i > j$, by Lemmas 1 and 3, we have

$$S(D_i, A_j) \xrightarrow[D_i, A_j]{}^* [D_i, A_j] = 2a_{ji}^{-2}\partial_j C_{ji} \xrightarrow[C_{ji}]{}^* 0.$$

When $i = j$, by Lemmas 1 and 3, we have

$$S(D_i, A_i) \xrightarrow[D_i, A_j]{}{}^{*} [D_i, A_i] = \sum_{l=1}^{i-1} 2a_{li}^{-2} \partial_l C_{li} \xrightarrow[C_{1i}, \ldots, C_{i-1i}]{}{}^{*} 0.$$

S-pair of D_i and B. When $i = 1$, the S-pair is $S(D_1, B) = \partial_1 B - \partial_1 B = 0$.
When $i > 1$, the initial monomials $\mathrm{in}_<(D_i) = \mathrm{in}_<(\partial_i)^3$, $\mathrm{in}_<(B) = \mathrm{in}_<(\partial_1)^2$ are coprime. By Lemmas 1 and 3, we have

$$S(D_i, B) \xrightarrow[D_i, B]{}{}^{*} [D_i, B] = 0.$$

S-pair of D_i and D_j. The initial monomials $\mathrm{in}_<(D_i) = \mathrm{in}_<(\partial_i)^3$, $\mathrm{in}_<(D_j) = \mathrm{in}_<(\partial_j)^3$ are coprime. By Lemmas 1 and 3, we have

$$S(D_i, D_j) \xrightarrow[D_i, D_j]{}{}^{*} [D_i, D_j] \xrightarrow[B, C_{1j}, \ldots, C_{ij}]{}{}^{*} 0.$$

S-pair of C_{ij} and D_k. The initial monomials are $\mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j)$, $\mathrm{in}_<(D_k) = \mathrm{in}_<(\partial_k)^3$. When $i \neq k$ and $j \neq k$, the initial monomials are coprime. By Lemmas 1 and 3, we have

$$S(C_{ij}, D_k) \xrightarrow[C_{ij}, D_k]{}{}^{*} [C_{ij}, D_k] \xrightarrow[C_{ik}, C_{jk}]{}{}^{*} 0.$$

When $i = k$, the initial monomials $\mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j), \mathrm{in}_<(D_i) = \mathrm{in}_<(\partial_i)^3$ are not coprime. This case needs a care of an order of applying reductions. We will reduce the S-pair by D_j and then reduce remainders by C_{ij}'s.

$$\begin{aligned}
S(C_{ij}, D_i) &= \partial_i^2 C_{ij} - a_{ij}\partial_j D_i \\
&= \partial_i^2 F_{ij} - a_{ij}\underline{\partial_i \partial_j^3} - a_{ij}\partial_j\Big(\sum_{l=i+1, l\neq j}^{n+1} \partial_i \partial_l^2 - \partial_i r^2 - \sum_{l=1}^{i-1} \partial_l a_{li}^{-1} F_{li}\Big) \\
&\xrightarrow[D_j]{}{}^{*} \partial_i^2 F_{ij} + a_{ij}\partial_i\Big(\sum_{l=j+1}^{n+1} \partial_j \partial_l^2 - \partial_j r^2 - \sum_{l=1}^{j-1} \partial_l a_{lj}^{-1} F_{lj}\Big) \\
&\qquad\qquad - a_{ij}\partial_j\Big(\sum_{l=i+1, l\neq j}^{n+1} \partial_i \partial_l^2 - \partial_i r^2 - \sum_{l=1}^{i-1} \partial_l a_{li}^{-1} F_{li}\Big) \\
&= -a_{ij}\partial_i(\partial_{i+1} a_{i+1j}^{-1} C_{i+1j} + \cdots + \partial_{j-1} a_{j-1j}^{-1} C_{j-1j}) \\
&\qquad\qquad - a_{ij}\partial_i\Big(\sum_{l=1}^{i-1} a_{lj}^{-1} F_{lj}\Big) + a_{ij}\partial_j\Big(\sum_{l=1}^{i-1} \partial_l a_{li}^{-1} F_{li}\Big) \\
&\xrightarrow[C_{i+1j}, \ldots, C_{j-1j}]{}{}^{*} -a_{ij}\partial_i\Big(\sum_{l=1}^{i-1} a_{lj}^{-1} F_{lj}\Big) + a_{ij}\partial_j\Big(\sum_{l=1}^{i-1} \partial_l a_{li}^{-1} F_{li}\Big) \\
&= -a_{ij} \sum_{l=1}^{i-1} (a_{lj}^{-1} \partial_l \partial_i F_{lj} - a_{li}^{-1} \partial_l \partial_j F_{li}).
\end{aligned}$$

Since $a_{lj}^{-1}\partial_l\partial_i F_{lj} - a_{li}^{-1}\partial_l\partial_j F_{li} \xrightarrow[C_{lj},C_{li},C_{ij}]{}{}^{*} 0$, the S-pair $S(C_{ij}, D_i)$ is reduced to 0.

When $j = k$, the initial monomials $\mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j), \mathrm{in}_<(D_j) = \mathrm{in}_<(\partial_j)^3$ are not coprime. This case also needs a care of an order of applying reductions.

$$
\begin{aligned}
S(C_{ij}, D_j) &= \partial_j^2 C_{ij} - a_{ij}\partial_i D_j \\
&= F_{ij}(\partial_i^2 + \partial_j^2) - a_{ij}\partial_i\Big(\sum_{l=j+1}^{n+1}\partial_j\partial_l^2 - \partial_j r^2 - \sum_{l=1,l\neq i}^{j-1}\partial_l a_{lj}^{-1}F_{lj}\Big) \\
&\xrightarrow[B]{}{}^{*} F_{ij}\Big(-\sum_{l=1,l\neq i,j}^{n+1}\partial_l^2 + r^2\Big) - a_{ij}\partial_i\Big(\sum_{l=j+1}^{n+1}\partial_j\partial_l^2 - \partial_j r^2 - \sum_{l=1,l\neq i}^{j-1}\partial_l a_{lj}^{-1}F_{lj}\Big) \\
&= F_{ij}\Big(-\sum_{l=1,l\neq i,j}^{n+1}\partial_l^2\Big) + r^2(a_{ij}\partial_i\partial_j + F_{ij}) - a_{ij}\partial_i\Big(\sum_{l=j+1}^{n+1}\partial_j\partial_l^2 - \sum_{l=1,l\neq i}^{j-1}\partial_l a_{lj}^{-1}F_{lj}\Big) \\
&\xrightarrow[C_{ij}]{}{}^{*} F_{ij}\Big(-\sum_{l=1,l\neq i,j}^{n+1}\partial_l^2\Big) - a_{ij}\partial_i\Big(\sum_{l=j+1}^{n+1}\partial_j\partial_l^2 - \sum_{l=1,l\neq i}^{j-1}\partial_l a_{lj}^{-1}F_{lj}\Big) \\
&= \Big(\sum_{l=j+1}^{n+1}\partial_l^2\Big)(-a_{ij}\partial_i\partial_j - F_{ij}) - F_{ij}\sum_{l=1,l\neq i}^{j-1}\partial_l^2 + a_{ij}\partial_i\sum_{l=1,l\neq i}^{j-1}\partial_l a_{lj}^{-1}F_{lj} \\
&\xrightarrow[C_{ij}]{}{}^{*} -F_{ij}\sum_{l=1,l\neq i}^{j-1}\partial_l^2 + a_{ij}\partial_i\sum_{l=1,l\neq i}^{j-1}\partial_l a_{lj}^{-1}F_{lj} \\
&= \sum_{l=1,l\neq i}^{j-1}\partial_l a_{ij}(-a_{ij}^{-1}\partial_l F_{ij} + a_{lj}^{-1}\partial_i F_{lj}).
\end{aligned}
$$

Since $-a_{ij}^{-1}\partial_l F_{ij} + a_{lj}^{-1}\partial_i F_{lj} \xrightarrow[C_{lj},C_{li},C_{ij}]{}{}^{*} 0$, the S-pair $S(C_{ij}, D_j)$ is reduced to 0.

S-pair of E and A_i. The initial monomials $\mathrm{in}_<(E) = \mathrm{in}_<(\partial_r), \mathrm{in}_<(A_i) = \mathrm{in}_<(\partial_{ii})$ are coprime. By Lemmas 1 and 4, we have

$$S(A_i, E) \xrightarrow[A_i,E]{}{}^{*} [A_i, E] = 0.$$

S-pair of E and B. The initial monomials $\mathrm{in}_<(E) = \mathrm{in}_<(\partial_r), \mathrm{in}_<(B) = \mathrm{in}_<(\partial_1)^2$ are coprime. By Lemmas 1 and 4, we have

$$S(B, E) \xrightarrow[B,E]{}{}^{*} [B, E] = -2B \xrightarrow[B]{}{}^{*} 0.$$

S-pair of E and C_{ij}. The initial monomials $\mathrm{in}_<(E) = \mathrm{in}_<(\partial_r), \mathrm{in}_<(C_{ij}) = \mathrm{in}_<(\partial_i)\mathrm{in}_<(\partial_j)$ are coprime. By Lemmas 1 and 4, we have

$$S(C_{ij}, E) \xrightarrow[C_{ij},E]{}{}^{*} [C_{ij}, E] = 0.$$

S-pair of D_i and E. The initial monomials $\mathrm{in}_<(E) = \mathrm{in}_<(\partial_r), \mathrm{in}_<(D_i) = \mathrm{in}_<(\partial_i)^3$ are coprime. By Lemmas 1 and 4, we have

$$S(D_i, E) \xrightarrow[D_i, E]{}^* [D_i, E] = -3D_i - 2\sum_{k=1}^{i-1} a_{ki}^{-1} \partial_k C_{ki} \xrightarrow[D_i, C_{1i}, \ldots, C_{i-1i}]{}^* 0.$$

We have proved that any S-pair is reduced to 0. By Buchberger's criterion, the set $\{A_i, B, C_{ij}, D_k, E\}$ is a Gröbner basis of $R\tilde{I}$. Q.E.D.

3 Gröbner Deformation of the Fisher-Bingham System

Consider the system of differential equations $I \cdot f = 0$. Intuitively speaking, we want to prove that the system I can be deformed to the diagonal system $\tilde{I}$ without increasing the holonomic rank. This can be done by a Gröbner basis computation with a weight vector $(-w, w)$ [6, Theorem 2.2.1]. However a straightforward calculation does not seem to be easy. We need to use some technical tricks to determine a suitable Gröbner deformation. Since these tricks may look too technical for the general n, we explain them in the case of $n = 1$ in Section 4 to clarify our idea without technical details of this section. Readers are expected to refer to the Section 4 when technicalities get complicated.

We will introduce new indeterminates to make our Gröbner basis computation possible by hand with employing the idea of the proof of [6, Theorem 3.1.3]. Let a_{pq}, b_i, c_i, d $(1 \le p \le q \le n+1,\ 1 \le i \le n+1)$ be constants, which we call slack variables when they are regarded as indeterminates. We put $g = \left(r^{d^3} \prod_{p \le q} x_{pq}^{a_{pq}^3} \right) f$ and make a change of the independent variables y_i by $y_i + b_i c_i$. Then, the system of differential equations for the function g is $I' \cdot g = 0$ where I' is the left ideal in D generated by the set of operators $G' = \{A'_{pq}, B, C'_{ij}, E'\}$ where

$$\begin{aligned}
A'_{pq} &= x_{pq}\partial_{pq} - x_{pq}\partial_p\partial_q - a_{pq}^3, \\
B &= \sum_{i=1}^{n+1} \partial_i^2 - r^2, \\
C'_{ij} &= x_{ij}\partial_i^2 + 2(x_{jj} - x_{ii})\partial_i\partial_j - x_{ij}\partial_j^2 \\
&\qquad + \sum_{s \ne i,j} (x_{sj}\partial_i\partial_s - x_{is}\partial_j\partial_s) + (y_j + b_j c_j)\partial_i - (y_i + b_i c_i)\partial_j, \\
E' &= r\partial_r - 2\sum_{i \le j} x_{ij}\partial_i\partial_j - \sum_{i=1}^{n+1} (y_i + b_i c_i)\partial_i - n - d^3.
\end{aligned}$$

The key fact is that the holonomic rank of I for f agrees with the holonomic rank of I' for g for any constants a_{pq}, b_i, c_i, d.

Let us make the same change of the variables for the diagonal system; let $\tilde{I}'$ be the left ideal of D generated by the set of operators $\tilde{G}' = \{\tilde{A}'_{ii}, B, \tilde{C}'_{ij}, \tilde{E}', x_{ij}\partial_{ij} - a^3_{ij}\ (i \neq j)\}$ where

$$\tilde{A}'_{ii} = x_{ii}\partial_i^2 - x_{ii}\partial_{ii} - a^3_{ii} \quad (1 \leq i \leq n+1),$$
$$B = \sum_{i=1}^{n+1} \partial_i^2 - r^2,$$
$$\tilde{C}'_{ij} = 2(x_{jj} - x_{ii})\partial_i\partial_j + (y_j + b_j c_j)\partial_i - (y_i + b_i c_i)\partial_j \quad (1 \leq i < j \leq n+1),$$
$$\tilde{E}' = r\partial_r - 2\sum_{i=1}^{n+1} x_{ii}\partial_i^2 - \sum_{i=1}^{n+1}(y_i + b_i c_i)\partial_i - n - d^3.$$

The holonomic ranks of $\tilde{I}$ and $\tilde{I}'$ agree.

Define the weight vector w by $w_{ij} = 1$, $(i \neq j)$, $w_{ii} = 0$, $w_k = 0$, and $w_r = 0$. Here, w_{ij} stands for ∂_{ij}, w_k stands for ∂_k, and w_r stands for ∂_r. The initial form $\mathrm{in}_{(-w,w)}(\ell)$, $\ell \in D$, is the sum of the highest $(-w,w)$-degree terms in ℓ and $\mathrm{in}_{(-w,w)}(I')$ is the left ideal generated by $\ell \in I'$ where the weight $-w$ stands for space variables x_{ij}, y_k, r corresponding to differential operators ∂_{ij}, ∂_k, ∂_r respectively (see, e.g., [6, Chapter 1]).

Theorem 3 *For generic complex numbers a_{pq}, b_i, c_i, d, we have*

$$\mathrm{in}_{(-w,w)}(I') = \tilde{I}'. \tag{2}$$

In order to prove the theorem, we regard a_{pq}, b_i, c_i, d as ring variables with the weight 0 and consider the following homogenized system of I':

$$A'^h_{pq} = h x_{pq}\partial_{pq} - x_{pq}\partial_p\partial_q - a^3_{pq},$$
$$B = \sum_{i=1}^{n+1} \partial_i^2 - r^2,$$
$$C'^h_{ij} = x_{ij}\partial_i^2 + 2(x_{jj} - x_{ii})\partial_i\partial_j - x_{ij}\partial_j^2 + \sum_{s\neq i,j}(x_{sj}\partial_i\partial_s - x_{is}\partial_j\partial_s) + (hy_j + b_j c_j)\partial_i - (hy_i + b_i c_i)\partial_j,$$
$$E'^h = hr\partial_r - 2\sum_{i\leq j} x_{ij}\partial_i\partial_j - \sum_{i=1}^{n+1}(hy_i + b_i c_i)\partial_i - nh^3 - d^3.$$

Let I'^h be a left $D^h[a,b,c,d]$-ideal generated by the set of operators $G'^h := \{A'^h_{pq}, B, C'^h_{ij}, E'^h\}$, where $D^h[a,b,c,d]$ is the homogenized Weyl algebra of the ring $D[a,b,c,d] = \mathbf{C}[a,b,c,d]\langle x_{ij}, y_k, r, \partial_{ij}, \partial_k, \partial_r\rangle$ with the homogenization variable h (see, e.g., [5, Section 9]). We introduce a new term order $<^h_{(-w,w,0)}$ over $D^h[a,b,c,d]$, which compares the total degree first, $(-w,w,0)$-degree second, otherwise we apply the following block order as a tie breaker: $d \gg r \gg \{a_{pq} \mid i \leq j\} \gg \{b_k\} \gg \{c_k\} \gg \{y_k\} \gg \partial_r \gg \{\partial_{ij} \mid i < j\} \gg \{\partial_{ii}\} \gg \{\partial_k\} \gg \{x_{ij} \mid$

$i < j\} \gg \{x_{ii}\} \gg h$. Here, the block $\{b_k\}$ has a lexicographic order so that $b_1 > b_2 > \cdots > b_{n+1}$. The use of this tie breaker is a key of our calculation. Although, the initial monomial $\mathrm{in}_{<^h_{(-w,w,0)}}(\ell)$ is an element of the associated commutative ring, we denote it by the associated element in $D^h[a,b,c,d]$ as long as no confusion arises. For example, we denote $\mathrm{in}_{<^h_{(-w,w,0)}}(\partial_{ij})$ by ∂_{ij} instead of ξ_{ij}.

Proposition 2 *A Gröbner basis of I'^h with respect to $<^h_{(-w,w,0)}$ is $G'^h = \{A'^h_{pq}, B, C'^h_{ij}, E'^h\}$.*

We need four lemmas for proving the proposition. These can be obtained by a straightforward calculation.

Lemma 5 *The initial monomials of the generators of I'^h are*

1. $\mathrm{in}_{<^h_{(-w,w,0)}}(A'^h_{pq}) = -a^3_{pq}$,
2. $\mathrm{in}_{<^h_{(-w,w,0)}}(B) = -r^2$,
3. $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ij}) = -b_i c_i \partial_j$,
4. $\mathrm{in}_{<^h_{(-w,w0)}}(E'^h) = -d^3$.

In particular, they are pairwise coprime expect in the case that the pair $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ij})$ and $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ik})$ and the pair $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ik})$ and $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{jk})$.

Lemma 6 *The commutators of two generators of I'^h are*

1. $[A'^h_{pq}, A'^h_{rs}] = [A'^h_{pq}, B] = [A'^h_{pq}, C'^h_{ij}] = [A'^h_{pq}, E'^h] = 0$ *(for any p,q,r,s,i,j)*,
2. $[B, C'^h_{ij}] = 0$, $[B, E'^h] = -2hB$,
3. $[C'^h_{ij}, C'^h_{kl}] = 0 \quad (\{i,j\} \cap \{k,l\} = \emptyset)$,
4. $[C'^h_{ij}, C'^h_{jk}] = C'^h_{ki} := -C'^h_{ik}$, $[C'^h_{ij}, C'^h_{ik}] = C'^h_{jk}$, $[C'^h_{ik}, C'^h_{jk}] = C'^h_{ij} \quad (i < j < k)$,
5. $[C'^h_{ij}, E'^h] = 0$.

Lemma 7 *The following holds for S-pairs of generators of I'^h:*

1. $S(A'^h_{pq}, A'^h_{rs}), S(A'^h_{pq}, B), S(A'^h_{pq}, C'^h_{ij}), S(A'^h_{pq}, E'^h) \longrightarrow^* 0$ *(for any p,q,r,s,i,j)*,
2. $S(B, C'^h_{ij}) \longrightarrow^* 0$, $S(B, E'^h) \longrightarrow^* 0$,
3. $S(C'^h_{ij}, C'^h_{kl}) \longrightarrow^* 0 \quad (\{i,j\} \cap \{k,l\} = \emptyset)$,
4. $S(C'^h_{ij}, C'^h_{jk}) \longrightarrow^* 0 \quad (i < j < k)$,
5. $S(C'^h_{ij}, E'^h) \longrightarrow^* 0$.

Proof. Lemma 1 holds in the homogenized Weyl algebra $D^h[a,b,c,d]$ too. Therefore, these follow from Lemmas 5 and 6. Q.E.D.

Lemma 8 *We put* $\hat{C}'^h_{ij} := b_jc_j\partial_i - b_ic_i\partial_j$ *and* $\check{C}'^h_{ij} := C'^h_{ij} - \hat{C}'^h_{ij}$*. Then the following cyclic relations hold:*

1. $\partial_k\hat{C}'^h_{ij} + \partial_i\hat{C}'^h_{jk} + \partial_j\hat{C}'^h_{ki} = \hat{C}'^h_{ij}\partial_k + \hat{C}'^h_{jk}\partial_i + \hat{C}'^h_{ki}\partial_j = 0,$
2. $\partial_k\check{C}'^h_{ij} + \partial_i\check{C}'^h_{jk} + \partial_j\check{C}'^h_{ki} = \check{C}'^h_{ij}\partial_k + \check{C}'^h_{jk}\partial_i + \check{C}'^h_{ki}\partial_j = 0,$
3. $\partial_kC'^h_{ij} + \partial_iC'^h_{jk} + \partial_jC'^h_{ki} = C'^h_{ij}\partial_k + C'^h_{jk}\partial_i + C'^h_{ki}\partial_j = 0,$
4. $b_kc_k\hat{C}'^h_{ij} + b_ic_i\hat{C}'^h_{jk} + b_jc_j\hat{C}'^h_{ki} = \hat{C}'^h_{ij}b_kc_k + \hat{C}'^h_{jk}b_ic_i + \hat{C}'^h_{ki}b_jc_j = 0.$

Proof of Proposition 2. By Lemma 7, we only need to check that the S-pairs $S(C'^h_{ij}, C'^h_{ik})$ and $S(C'^h_{ik}, C'^h_{jk})$ are reduced to zero.

The former is $S(C'^h_{ij}, C'^h_{ik}) = \partial_kC'^h_{ij} - \partial_jC'^h_{ik} = \partial_kC'^h_{ij} + \partial_jC'^h_{ki}$, because the initial monomials are $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ij}) = -b_ic_i\partial_j$ and $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ik}) = -b_ic_i\partial_k$. The S-pair is equal to $-\partial_iC'^h_{jk}$ by the 3rd formula of Lemma 8. This implies that it is reduced to zero.

The latter is $S(C'^h_{ik}, C'^h_{jk}) = b_jc_jC'^h_{ik} - b_ic_iC'^h_{jk}$, because the initial monomials are $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{ik}) = -b_ic_i\partial_k$ and $\mathrm{in}_{<^h_{(-w,w,0)}}(C'^h_{jk}) = -b_jc_j\partial_k$.

Firstly, we show that it can be expressed as

$$\begin{aligned} S(C'^h_{ik}, C'^h_{jk}) &= \left(\sum_{s=1}^{n+1}(\delta_{ks}+1)x_{ks}\partial_s + hy_k + b_kc_k\right)C'^h_{ij} \\ &+ \left(\sum_{s=1}^{n+1}(\delta_{is}+1)x_{is}\partial_s + hy_i\right)C'^h_{jk} \\ &+ \left(\sum_{s=1}^{n+1}(\delta_{js}+1)x_{js}\partial_s + hy_j\right)C'^h_{ki}. \end{aligned} \tag{3}$$

Here, the symbol δ is the Kronecker delta. This expression of the S-pair is a key of our proof. Since each monomial of the left hand side (LHS in short) has at least one variable in b_i, b_j or b_k, we divide the right hand side (RHS in short) into two parts: S_1 contains these variables, S_2 does not contain them. Then,

$$\begin{aligned} S_2 = &\left(\sum_{s=1}^{n+1}(\delta_{ks}+1)x_{ks}\partial_s + hy_k\right)\check{C}'^h_{ij} + \left(\sum_{s=1}^{n+1}(\delta_{is}+1)x_{is}\partial_s + hy_i\right)\check{C}'^h_{jk} \\ &+ \left(\sum_{s=1}^{n+1}(\delta_{js}+1)x_{js}\partial_s + hy_j\right)\check{C}'^h_{ki} \end{aligned}$$

is equal to zero by a straightforward calculation and we have

$$\begin{aligned}
S_1 &= b_k c_k C'^h_{ij} + \left(\sum_{s=1}^{n+1}(\delta_{ks}+1)x_{ks}\partial_s + hy_k\right)\hat{C}'^h_{ij} \\
&\quad + \left(\sum_{s=1}^{n+1}(\delta_{is}+1)x_{is}\partial_s + hy_i\right)\hat{C}'^h_{jk} + \left(\sum_{s=1}^{n+1}(\delta_{js}+1)x_{js}\partial_s + hy_j\right)\hat{C}'^h_{ki} \\
&= b_k c_k C'^h_{ij} \\
&+ b_j c_j \left(\sum_{s=1}^{n+1}(\delta_{ks}+1)x_{ks}\partial_s\partial_i + hy_k\partial_i\right) - b_i c_i \left(\sum_{s=1}^{n+1}(\delta_{ks}+1)x_{ks}\partial_s\partial_j + hy_k\partial_j\right) \\
&+ b_k c_k \left(\sum_{s=1}^{n+1}(\delta_{is}+1)x_{is}\partial_s\partial_j + hy_i\partial_j\right) - b_j c_j \left(\sum_{s=1}^{n+1}(\delta_{is}+1)x_{is}\partial_s\partial_k + hy_i\partial_k\right) \\
&+ b_i c_i \left(\sum_{s=1}^{n+1}(\delta_{js}+1)x_{js}\partial_s\partial_k + hy_j\partial_k\right) - b_k c_k \left(\sum_{s=1}^{n+1}(\delta_{js}+1)x_{js}\partial_s\partial_i + hy_j\partial_i\right) \\
&= b_j c_j C'^h_{ik} - b_i c_i C'^h_{jk} = S(C'^h_{ik}, C'^h_{jk}).
\end{aligned}$$

Finally, we show that the expression (3) is a standard representation. We note that the RHS of (3) has three terms, which appear in the first, in the second and in the third lines of (3) respectively. We may show that the initial monomial of the LHS is no less than the initial monomials of the three terms of the RHS with respect to the term order $<^h_{(-w,w,0)}$. The initial monomial of the LHS $b_i c_i b_k c_k \partial_j$ coincides with that of the 1st term of the RHS. Moreover, all monomials appearing in the 2nd and the 3rd term of the RHS have the same total degree 5 and $(-w, w, 0)$-degree 0, and they have a degree at most one with respect to b_i, b_j and b_k. This implies that they are less than $b_i c_i b_k c_k \partial_j$. Thus, we have shown that the expression is a standard representation and the S-pair is reduced to zero.

Since all S-pairs are reduced to zero, the conclusion follows from Buchberger's criterion. Q.E.D.

Proof of Theorem 3. Let $I'(a, b, c, d)$ be the left ideal generated by G' in the ring $D[a, b, c, d]$. It follows from Proposition 2 and [6, Theorem 1.2.4] that $G'^h|_{h=1} = G'$ is a Gröbner basis of $I'(a, b, c, d)$ with respect to $<_{(-w,w,0)}$. Therefore, we have

$$\mathrm{in}_{<_{(-w,w,0)}}(I'(a, b, c, d)) = \langle \mathrm{in}_{<_{(-w,w,0)}}(G')\rangle = \langle \tilde{G}'\rangle \quad \text{in } D[a, b, c, d].$$

We may replace $D[a, b, c, d]$ by $D(a, b, c, d) = \mathbf{C}(a, b, c, d)\langle x_{ij}, y_k, r, \partial_{ij}, \partial_k, \partial_r\rangle$. Here, $\mathbf{C}(a, b, c, d)$ is the rational function field with variables $a = (a_{pq}), b = (b_i), c = (c_i)$ and d. Then, the following holds:

$$\mathrm{in}_{(-w,w)}(I'(a, b, c, d)) = \langle \tilde{G}'\rangle \quad \text{in } D(a, b, c, d).$$

This implies the conclusion. Q.E.D.

Proof of the Main theorem 1. It follows from (2) that rank $(I) \geq$ rank $(\mathrm{in}_{(-w,w)}(I'))$ by Theorem 2.2.1 in [6]. Therefore, we have rank $(I) \geq 2n+2$ by Theorem 2. The opposite inequality follows from Theorem 3 of [4]. Q.E.D.

4 A Proof in the case of $n = 1$

In order to clarify ideas of the proof in Section 3, we present a proof of our theorem in the case of $n = 1$.

The 1-dimensional Fisher-Bingham system of differential equations $I \subset \mathbf{C}\langle x_{11}, x_{12}, x_{22}, y_1, y_2, r, \partial_{11}, \partial_{12}, \partial_{22}, \partial_1, \partial_2, \partial_r\rangle$ is

$$\begin{aligned} I = \langle & A_{11} = \partial_{11} - \partial_1^2,\ A_{12} = \partial_{12} - \partial_1\partial_2, \\ & A_{22} = \partial_{22} - \partial_2^2, \\ & B = \partial_1^2 + \partial_2^2 - r^2, \\ & C_{12} = x_{12}\partial_1^2 + 2(x_{22} - x_{11})\partial_1\partial_2 - x_{12}\partial_2^2 + y_2\partial_1 - y_1\partial_2, \\ & E = r\partial_r - 2(x_{11}\partial_1^2 + x_{12}\partial_1\partial_2 + x_{22}\partial_2^2) - (y_1\partial_1 + y_2\partial_2) - 1\rangle. \end{aligned}$$

The upper bound of rank(I) is $2 \cdot 1 + 2 = 4$ as given in [4, Theorem 3]. We will show the lower bound of the rank(I) is 4 by using the following general inequality [6, Theorem 2.2.1]:

$$\mathrm{rank}(I) \geq \mathrm{in}_{(-w,w)}(I).$$

Firstly, we make some change of variables, because it seems to be difficult to calculate $\mathrm{in}_{(-w,w)}(I)$ directly. Let $a_{11}, a_{12}, a_{22}, b_1, b_2, c_1, c_2, d$ be constants, which will be used as slack variables. We put $g = r^{d^3} x_{11}^{a_{11}^3} x_{12}^{a_{12}^3} x_{22}^{a_{22}^3} f$ where the function f is a solution of the system of differential equations $I \cdot f = 0$. Moreover, we make a change of variables y_1, y_2 by $y_1 + b_1c_1, y_2 + b_2c_2$ respectively. Then, the system of differential equations for g is given by $I' \cdot g = 0$, where

$$\begin{aligned} I' = \langle & A'_{11} = x_{11}\partial_{11} - x_{11}\partial_1^2 - a_{11}^3,\ A'_{12} = x_{12}\partial_{12} - x_{12}\partial_1\partial_2 - a_{12}^3, \\ & A'_{22} = x_{22}\partial_{22} - x_{22}\partial_2^2 - a_{22}^3, \\ & B = \partial_1^2 + \partial_2^2 - r^2, \\ & C'_{12} = x_{12}\partial_1^2 + 2(x_{22} - x_{11})\partial_1\partial_2 - x_{12}\partial_2^2 + (y_2 + b_2c_2)\partial_1 - (y_1 + b_1c_1)\partial_2, \\ & E' = r\partial_r - 2(x_{11}\partial_1^2 + x_{12}\partial_1\partial_2 + x_{22}\partial_2^2) \\ & \qquad - ((y_1 + b_1c_1)\partial_1 + (y_2 + b_2c_2)\partial_2) - 1 - d^3\rangle. \end{aligned}$$

We note that rank$(I) =$ rank(I') holds for any set of values of the constants.

Secondly, we calculate $\mathrm{in}_{(-w,w)}(I')$ for the weight vector $w = (0, 1, 0, 0, 0, 0)$ where each weight stands for the variables $\partial_{11}, \partial_{12}, \partial_{22}, \partial_1, \partial_2, \partial_r$ respectively. In order to perform Buchberger's algorithm with respect to the $(-w, w)$-weight

order, we need to consider the homogenized system I'^h for I':

$$
\begin{aligned}
I'^h = \langle A'^h_{11} &= hx_{11}\partial_{11} - x_{11}\partial_1^2 - \underline{a_{11}^3},\ A'^h_{12} = hx_{12}\partial_{12} - x_{12}\partial_1\partial_2 - \underline{a_{12}^3},\\
A'^h_{22} &= hx_{22}\partial_{22} - x_{22}\partial_2^2 - \underline{a_{22}^3},\\
B &= \partial_1^2 + \partial_2^2 - \underline{r^2},\\
C'^h_{12} &= x_{12}\partial_1^2 + 2(x_{22} - x_{11})\partial_1\partial_2 - x_{12}\partial_2^2 + (y_2h + b_2c_2)\partial_1 - (y_1h + \underline{b_1c_1)\partial_2},\\
E'^h &= hr\partial_r - 2(x_{11}\partial_1^2 + x_{12}\partial_1\partial_2 + x_{22}\partial_2^2)\\
&\qquad - ((hy_1 + b_1c_1)\partial_1 + (hy_2 + b_2c_2)\partial_2) - h^3 - \underline{d^3}\rangle.
\end{aligned}
$$

We denote by $<^h_{(-w,w,0)}$ an order in the homogenized Weyl algebra which compares the total degree first, $(-w,w,0)$-degree second, otherwise we apply the following block order as a tie breaker: $d \gg r \gg \{a_{11}, a_{12}, a_{22}\} \gg \{b_1 > b_2\} \gg \{c_1, c_2\} \gg \{y_1, y_2\} \gg \partial_r \gg \{\partial_{12}\} \gg \{\partial_{11}, \partial_{22}\} \gg \{\partial_1, \partial_2\} \gg \{x_{12}\} \gg \{x_{11}, x_{22}\} \gg h$. Here, the symbol $>$ represents the lexicographic order. The underlined parts in I'^h are initial terms with respect to $<^h_{(-w,w,0)}$. They are pairwise coprime and their commutators are equal to zero except $[B, E'^h] = -2hB$. From Lemma 1, we conclude that the set $\{A'^h_{11}, A'^h_{12}, A'^h_{22}, B, C'^h_{12}, E'^h\}$ is a Gröbner basis of I'^h in $D^h[a_{11}, a_{12}, a_{22}, b_1, b_2, c_1, c_2, d]$ with respect to $<^h_{(-w,w,0)}$. In other words, the transformation of dependent and independent variables gives us the Gröbner basis without adding new elements. Dehomogenizing I'^h, we obtain

$$
\begin{aligned}
\mathrm{in}_{(-w,w)}(I') = \langle \tilde{A}'_{11} &= x_{11}\partial_{11} - x_{11}\partial_1^2 - a_{11}^3,\ \tilde{A}'_{12} = x_{12}\partial_{12} - a_{12}^3,\\
\tilde{A}'_{22} &= x_{22}\partial_{22} - x_{22}\partial_2^2 - a_{22}^3,\\
B &= \partial_1^2 + \partial_2^2 - r^2,\\
\tilde{C}'_{12} &= 2(x_{22} - x_{11})\partial_1\partial_2 + (y_2 + b_2c_2)\partial_1 - (y_1 + b_1c_1)\partial_2,\\
\tilde{E}' &= r\partial_r - 2(x_{11}\partial_1^2 + x_{22}\partial_2^2)\\
&\qquad - ((y_1 + b_1c_1)\partial_1 + (y_2 + b_2c_2)\partial_2) - 1 - d^3\rangle.
\end{aligned}
$$

In this calculation, we regard $a_{11}, a_{12}, a_{22}, b_1, b_2, c_1, c_2, d$ as ring variables with the weight 0. As in the proof of Theorem 3, the equation above holds when $a_{11}, \ldots, d$ are specialized to generic complex numbers. The holonomic rank of the $(-w,w)$-initial ideal $\tilde{I}' := \mathrm{in}_{(-w,w)}(I')$ coincides with that of the diagonal system transformed by the same change of variables for I' from I. The holonomic rank of $\tilde{I}'$ agrees with that of

$$
\begin{aligned}
\tilde{I} = \langle \tilde{A}_{11} &= \underline{\partial_{11}} - \partial_1^2,\ \tilde{A}_{12} = \underline{\partial_{12}},\\
\tilde{A}_{22} &= \underline{\partial_{22}} - \partial_2^2,\\
B &= \underline{\partial_1^2} + \partial_2^2 - r^2,\\
\tilde{C}_{12} &= 2(x_{22} - x_{11})\underline{\partial_1\partial_2} + y_2\partial_1 - y_1\partial_2,\\
\tilde{E} &= r\underline{\partial_r} - 2(x_{11}\partial_1^2 + x_{22}\partial_2^2) - (y_1\partial_1 + y_2\partial_2) - 1\rangle.
\end{aligned}
$$

Hence, we obtain the following inequality:

$$\mathrm{rank}(I) = \mathrm{rank}(I') \geq \mathrm{rank}(\mathrm{in}_{(-w,w)}(I')) = \mathrm{rank}(\tilde{I}') = \mathrm{rank}(\tilde{I}).$$

Finally, we show that $\mathrm{rank}(\tilde{I}) = 4$. Proposition 1 tells us that the set

$$\begin{aligned}&\{\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{22}, B, \tilde{C}_{12}, \tilde{E},\\ &\quad D_2 = 2(x_{22} - x_{11}) \left(\underline{\partial_2^3} - r^2 \partial_2 - \frac{y_2 \partial_1^2 - y_1 \partial_1 \partial_2 - \partial_2}{2(x_{22} - x_{11})} \right) \Bigg\}.\end{aligned}$$

is a Gröbner basis of $R\tilde{I}$ with respect to the block order $\{\partial_r\} \gg \{\partial_{11} > \partial_{22}\} \gg \{\partial_1 > \partial_2\}$, where the tie breaker $>$ represents the graded lexicographic order. The set of standard monomials is $\{1, \partial_1, \partial_2, \partial_2^2\}$. It means that the holonomic rank of $\tilde{I}$ is 4.

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